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# A novel strange attractor with a stretched loop

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**Abstract** This short paper introduces a new 3D strange attractor topologically different from any other known chaotic attractors. The intentionally constructed model of three autonomous first-order differential equations derives from the coupling-induced complexity of the well-established 2D Lotka–Volterra oscillator. Its chaotification process via an anti-equilibrium feedback allows the exploration of a new domain of dynamical behavior including chaotic patterns. To focus a rapid presentation, a fixed set of parameters is selected linked to the widest range of dynamics. Indeed, the new system leads to a chaotic attractor exhibiting a double scroll bridged by a loop. It mutates to a single scroll with a very stretched loop by the variation of one parameter. Indexes of stability of the equilibrium points corresponding to the two typical strange attractors are also investigated. To encompass the global behavior of the new low-dimensional dissipative dynamical model, diagrams of bifurcation displaying chaotic bubbles and windows of periodic oscillations are computed. Besides, the dominant exponent of the Lyapunov spectrum is positive reporting the chaotic nature of the system. Eventually, the novel chaotic model is suitable for digital signal encryption in the field of communication with a rich set of keys.

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## 1 Introduction

In a seminal paper, Lorenz [1] found the first chaotic attractor in a three-dimensional autonomous system while studying atmospheric instability. Its relatively simple nonlinear structure displays both sophisticated and stable dynamical behaviors.

Followed by Rössler [2], Chua [3, 4], Sprott [5], etc., chaos from nonlinear approach has been shown to be useful to overlap classical results in a global framework where strange chaotic patterns are possible conjectures. Yet, chaotic behavior probably exists and deserves a deep exploration. Indeed, in a wide range of disciplines, discovering chaos in low-dimensional dissipative dynamical systems stimulates research projects to redefine models from linear to nonlinear specifications.

Purposefully creating chaos can be a nontrivial task to focus a new kind of dynamical patterns as shown by several chaotic models recently discovered [6–9]. In addition, process of chaotification or anti-control of chaos (see, for example, [10–13], and references therein) explores the induced chaotic behaviors of an originally nonchaotic system via different techniques. To this end, and derived from this approach, we applied previously this methodology to the Van der Pol

2D oscillator leading to a specific class of 3D strange attractors [14] implemented eventually in an electronic circuit [15].

The present research aims the foundation of another canonical class of chaotic systems, according the coupling-induced complexity approach and, at best of our knowledge, with a new topological structure.

## 2 The modelization

Methodology of the feedback anti-control constitutes a "bridge" to reach chaos. It generates complexity triggering a mechanism of new dynamical patterns from elementary 2D models.

Indeed, we retain, to chaotify, the relatively simple and well-known Lotka–Volterra 2D system, modelizing the interspecific competition between two species [16–18].

### 2.1 A Lotka–Volterra-like system

Disregarding the peculiarity of the 2D Lotka–Volterra system, we focus only on its dynamical behavior expecting chaotic pattern through the linkage of an endogenous perturbation into a third dimension. Firstly, a quadratic term of $x$ is introduced in the $y$-equation of the Lotka–Volterra oscillator to extend its domain of variation:

$$dx/dt = x(a - y)$$
$$dy/dt = -y\big(b - x^2\big)$$

where $x$ and $y$ are the state variables of the model, and $a$ and $b$ positive scalars.

This Lotka–Volterra-like System (LVLS) leads to two stable and symmetric limit-cycles in the positive domain of $y$, depending strongly on the initial conditions.

The extension to the third dimension of the LVLS far from its "natural" oscillation could be obtained by injecting perturbation through the $z$-equation stressing the "robustness" of the system. One would expect the emergence of complexity and chaotic patterns in our application from this connection.

### 2.2 A new chaotic system

The exploration of new dynamical behaviors leads us to connect a $z$-equation to the LVLS. Its formulation, chosen between several specifications, is controlled by the parameter $\alpha$.

We obtain a new system governed by the following three-nonlinear differential equations:

$$dx/dt = x(a - y) + \alpha z$$
$$dy/dt = -y\big(b - x^2\big)$$
$$dz/dt = -x(c - sz) - \beta z$$

where $x$, $y$, and $z$ are the state variables of the system, and $a$, $\alpha$, $b$, $c$, $s$, and $\beta$ are the parameters.

The model embeds two quadratic nonlinearities and only one cubic term, respectively, $xy$, $xz$, and $yx^2$. Requirement of short analysis lead us to fix the set of parameters to $P(a, \alpha, b, c, \beta) = (4, 0.3, 1, 1.5, 0.05)$, and therefore s becomes the "controller" of the system.

For $s_0 = 1$, the obtained phase portrait displays a chaotic attractor with double scroll connected by a singular loop (Fig. 1). Changing the control parameter to $s_1 = 6$, the attractor appears to be different, exhibiting a single wing with a very stretched loop (Fig. 2).

The equilibrium points of these two attractors are found by setting

$$dx/dt = dy/dt = dz/dt = 0$$

For the specification of parameters $P$, and $s_0 = 1$, we obtain by solving the second equation, $y = 0$ or $x = \pm 1$. While $y = 0$, its substitution into the first equation gives $E_0(x_0, y_0, z_0)$ and $E_1(x_1, y_1, z_1)$. When $x = \pm 1$, the two other equilibria are namely $E_2(x_2, y_2, z_2)$ and $E_3(x_3, y_3, z_3)$.

The elementary attributes of these equilibria given by the corresponding eigenvalues $\lambda_i$ are found by solving the characteristic equation

$$|J - \lambda I| = 0$$

where $I$ is the unit matrix, and $J$ is the Jacobian of the model. In Table 1, coordinates, eigenvalues and features of stability related to the equilibria are reported. Similar characteristics are recorded for $P$ with $s_1 = 6$ in Table 2.

**Table 1** Equilibria of the system for $s_0 = 1$

| Coordinates of the equilibia | The corresponding characteristic equation, $\|J - \lambda I\| = 0$, and eigenvalues | Index$^{(1)}$ and stability |
|---|---|---|
| $E_0(x_0, y_0, z_0) = (0, 0, 0)$ | $\lambda^3 - 2.95\lambda^2 - 5.50\lambda - 1.55 = 0$<br>$\lambda_1 \approx 4.270$<br>$\lambda_2 \approx -1.028$<br>$\lambda_3 \approx -0.341$ | Index-1<br>Unstable:<br>*Saddle point* |
| $E_1(x_1, y_1, z_1) = (0.062, 0, 0.833)$ | $\lambda^3 - 3.01\lambda^2 - 5.98\lambda - 1.90 = 0$<br>$\lambda_1 \approx 4.428$<br>$\lambda_2 \approx -1$<br>$\lambda_3 \approx -0.428$ | Index-1<br>Unstable:<br>*Saddle point* |
| $E_2(x_2, y_2, z_2) = (1, 4.473, 1.578)$ | $\lambda^3 - 0.47\lambda^2 + 8.52\lambda - 8.49 = 0$<br>$\lambda_1 \approx 0.931$<br>$\lambda_2 \approx -0.265 + 2.99i$<br>$\lambda_3 \approx -0.265 - 2.99i$ | Index-1<br>Unstable:<br>*Spiral saddle point* |
| $E_3(x_3, y_3, z_3) = (-1, 3.571, 1.428)$ | $\lambda^3 + 0.621\lambda^2 + 7.11\lambda + 7.49 = 0$<br>$\lambda_1 \approx -1$<br>$\lambda_2 \approx 0.19 + 2.730i$<br>$\lambda_3 \approx 0.19 - 2.730i$ | Index-2<br>Unstable:<br>*Spiral saddle point* |

(1) Index reports the number of eigenvalues with real parts $\mathrm{Re}(\lambda) > 0$. From 1 to 3, it indicates the degree of instability. Index-0: null or negative real parts of all eigenvalues of the equilibrium characterize its stability

**Table 2** Equilibria of the system for $s_1 = 6$

| Coordinates of the equilibia | The corresponding characteristic equation, $\|J - \lambda I\| = 0$, and eigenvalues | Index$^{(1)}$ and stability |
|---|---|---|
| $E_0(x_0, y_0, z_0) = (0, 0, 0)$ | $\lambda^3 - 2.95\lambda^2 - 3.75\lambda + 0.45 = 0$<br>$\lambda_1 \approx 3.888$<br>$\lambda_2 \approx -1.037$<br>$\lambda_3 \approx 0.099$ | Index-2<br>Unstable:<br>*Saddle point* |
| $E_1(x_1, y_1, z_1) = (0.010, 0, 0.138)$ | $\lambda^3 - 2.99\lambda^2 + 4.16\lambda + 0.20 = 0$<br>$\lambda_1 \approx -0.046$<br>$\lambda_2 \approx 1.473 + 1.458i$<br>$\lambda_3 \approx 1.473 - 1.458i$ | Index-2<br>Unstable:<br>*Spiral saddle point* |
| $E_2(x_2, y_2, z_2) = (1, 4.075, 0.252)$ | $\lambda^3 + 5.2\lambda^2 - 3.69\lambda - 48.49 = 0$<br>$\lambda_1 \approx 2.716$<br>$\lambda_2 \approx -3.958 + 1.465i$<br>$\lambda_3 \approx -3.958 - 1.465i$ | Index-1<br>Unstable:<br>*Spiral saddle point* |
| $E_3(x_3, y_3, z_3) = (-1, 3.925, 0.248)$ | $\lambda^3 - 5.97\lambda^2 - 8.30\lambda - 47.48 = 0$<br>$\lambda_1 \approx 2.849$<br>$\lambda_2 \approx -5.957$<br>$\lambda_3 \approx -2.791$ | Index-1<br>Unstable:<br>*Saddle point* |

## 3 Global behavior

In order to detect the dynamical transition between the two topologically different strange attractors (Figs. 1 and 2), a diagram of bifurcation is computed by varying the parameter s from its null value (Fig. 3a). One can observe different dynamical patterns, within chaotic bubbles and several windows of stability.

As it can be seen in Fig. 3b, the two first windows of stability display cascades of period-adding bifur-

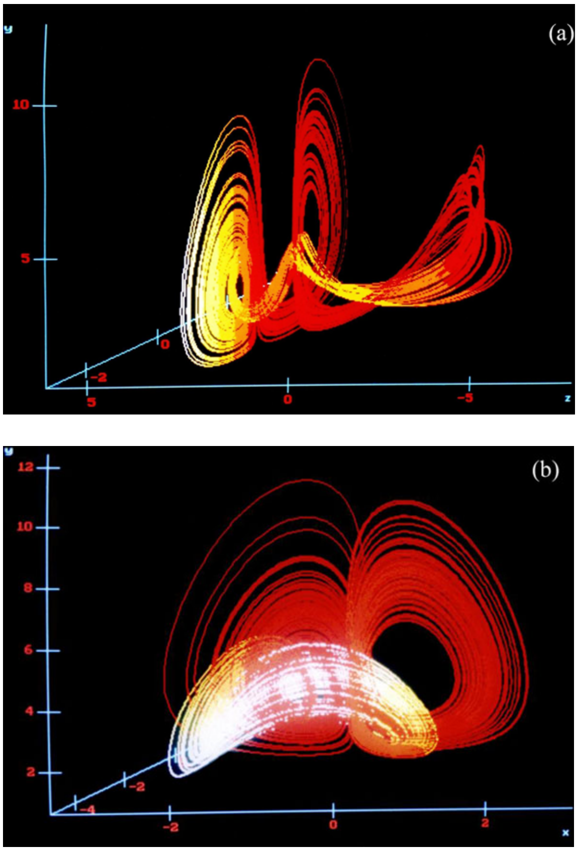


**Fig. 1** Phase portrait of the new chaotic attractor. It exhibits a double-scroll bridged by a stretched loop for $s_0 = 1$. (**a**) *Left* and (**b**) *front images*

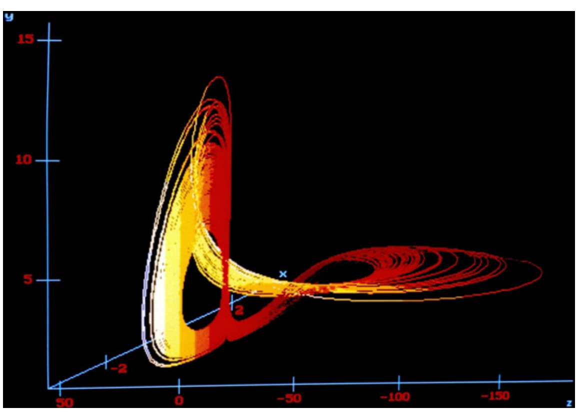

**Fig. 2** An evolved single-left scroll attractor. The scroll is connected to an orthogonally very stretched loop for $s_1 = 6$

cations. However, beyond a threshold (approximately $s = 0.65$), this process is reversed in the next cascade, through a period-decreasing bifurcation phenomenon

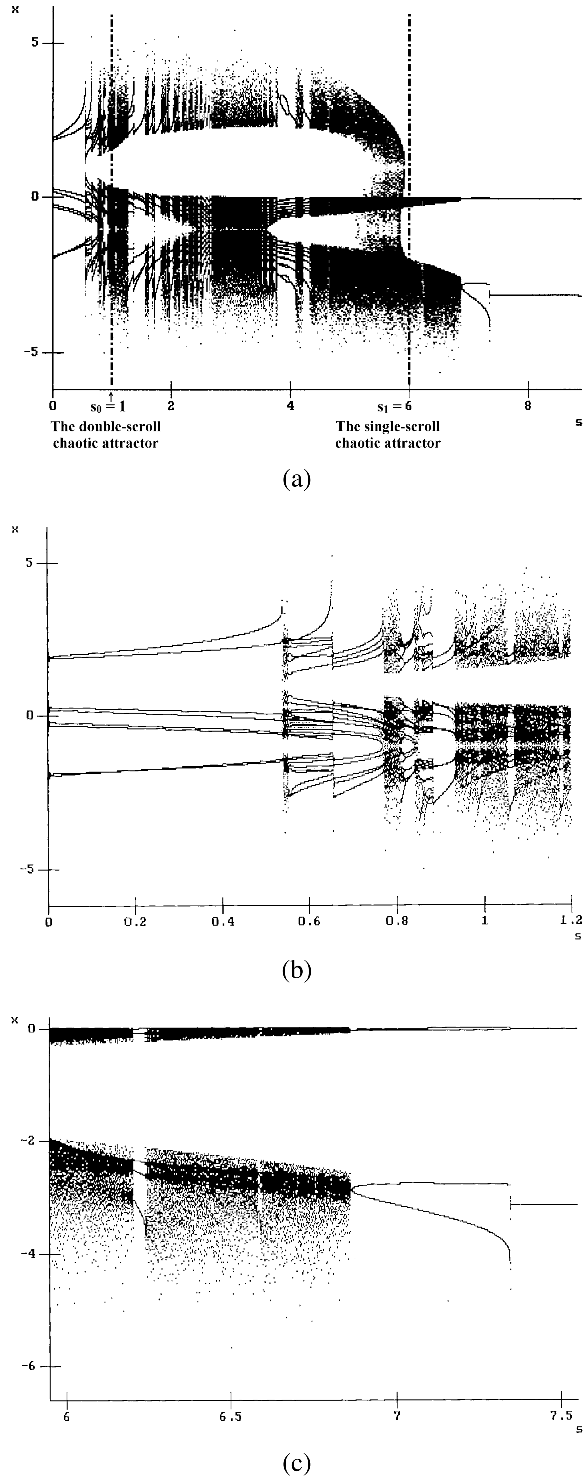


**Fig. 3** Diagram of bifurcation. *Dots* are related to both scroll and loop for the trajectories cutting the plane $(x, z)$ at $y = 5$. (**a**) The bubble of chaos, containing the two strange attractors, appears with several immersed windows of stability. (**b**) The first and (**c**) the final stages of bifurcations

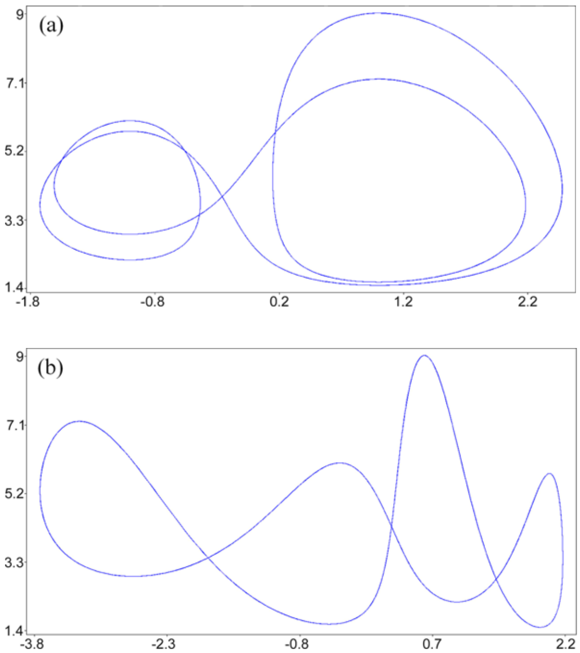


**Fig. 4** Projections of the limit-cycle for $s_3 = 0.4$. (**a**) $x$–$y$ phase plane, (**b**) $y$–$z$ phase plane

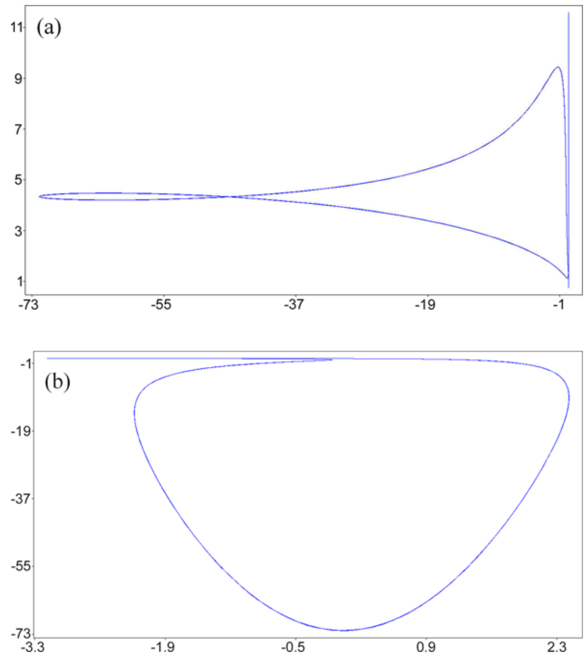


**Fig. 5** Projections of the limit-cycle for $s_4 = 5.8$. (**a**) $x$–$y$ phase plane, (**b**) $y$–$z$ phase plane

until the final period-2 motion (Fig. 3c). Inside the immersed windows of regular cycles, the periodic orbit of $s_3 = 0.4$ is observed (Fig. 4). We notice its shape very close to the chaotic attractor shown in Fig. 1. However, for the value $s_4 = 5.8$, the displayed limit-cycle (Fig. 5) is related to the chaotic attractor shown in Fig. 2.

Besides, as we are interested in the appearance of chaotic dynamics of the system, we focus on the parameter domain stressing the robustness of the LVLS. It would be useful to consider $\alpha$, the control parameter of the perturbation intensity, to check this feature. To this end, we compute the associated diagram of bifurcation with the set of parameters $Q(a, b, c, s, \beta) = (4, 1, 1.5, 1, 0.05)$, from the null value of $\alpha$ when LVLS and $z$-equation are disconnected (Fig. 6).

We notice that the system reaches a chaotic pattern immediately from the lowest value of $\alpha$ (stepsize $= 10^{-5}$). The robustness of the LVLS forced by the anti-equilibrium feedback $z$ appears deeply limited. However, the chaos vanishes in a final period-4 orbit.

Eventually, does the new system have explicitly and globally a chaotic nature? It is known that the spectrum of Lyapunov exponents is the most useful diagnostic to quantify chaos. When the nearby trajectories in the phase space diverge at exponential rates, giving a positive Lyapunov exponent, the dynamics become unpredictable. Any system containing at least one positive Lyapunov exponent is defined to be chaotic. As can be seen from the Lyapunov spectrum for a vary-

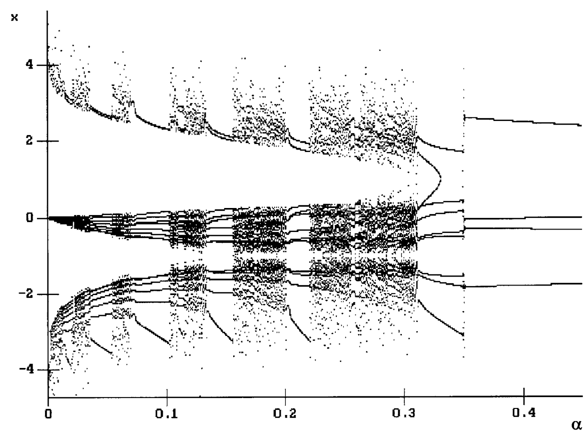


**Fig. 6** Robustness of the LVLS. Set of parameters, $Q(a, b, c, s, b) = (4, 1, 1.5, 1, 0.05)$. The diagram of bifurcation indicates that weak values of $\alpha$ create chaotic patterns

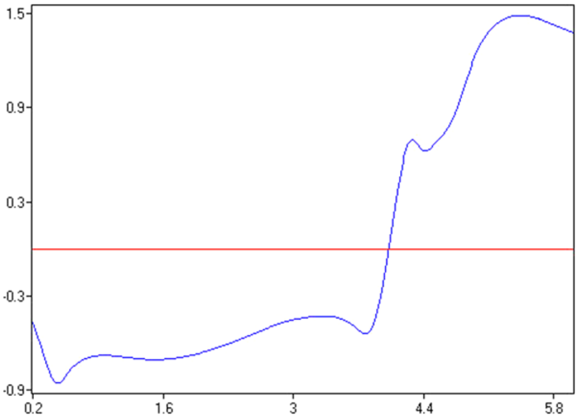


**Fig. 7** Largest Lyapunov exponent for the specification $P$ and $s$ varying in the range $[0.2, 6.2]$

ing parameter $s$ in the range $[0.2, 6.2]$, the dominant exponent reaches positive value (Fig. 7).

## 4 Concluding remarks

Creating chaos through a relatively simple algebraic structure by chaotifying an originally stable oscillator, the LVLS in the present case, is still a challenging task. The new chaotic system depicts a complex 2-scroll butterfly-shaped attractor exhibiting a singular loop. It mutates to a single wing, with a very stretched orthogonally positioned loop, according to its sensitive dependency on parameters. Several specifications of the feedback $z$-equation have been experimented. The simplest one, inducing the widest range of dynamical behaviors, has been selected.

This intentionally constructed chaotic system does not reincarnate known strange attractors. Indeed, at the best of our knowledge, the appearance of the new chaotic attractors is utterly different from the other existing chaotic systems showing double or a single scroll (Lorenz-like systems, Rössler, Chua, Sprott, etc.). It can be verified that there does not exist diffeomorphism between the new system and the previous chaotic 3D models since their eigenvalue structure is not equivalent to their corresponding equilibrium points.

Eventually, the new system could be suitable for digital signal encryption in the communication field, and its variants provide a very large set of encryption keys. The complete scheme of its electronic implementation can be easily designed using the State Controlled Cellular Network Nets paradigm. It simplifies the experimental realization of the chaotic circuits reducing significantly the operational amplifiers (opamps) compounds of the cell arrays. Indeed, Arena et al. [19] have previously proven the suitability of the framework for $n$-scroll chaotic oscillators.

Further developments and extended analysis related to the set of parameters will be investigated in a future work.

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## References

1. Lorenz, E.N.: Deterministic nonperiodic flow. J. Atmos. Sci. **20**, 130–141 (1963)
2. Rössler, O.E.: Continuous chaos; four prototype equations. Ann. N.Y. Acad. Sci. **316**, 376–392 (1979)
3. Matsumuto, T.: A chaotic attractor from Chua's circuit. IEEE Trans. Circuits Syst. **31**(12), 1055–1058 (1984)
4. Chua, L.O.: The genesis of Chua's circuit. AEÜ, Int. J. Electron. Commun. **46**(4), 250–257 (1992)
5. Sprott, J.C.: Some simple chaotic flows. Phys. Rev. E **50**, R647–R650 (1994)
6. Wang, L.: 3-scroll and 4-scroll chaotic attractors generated from a new 3-D quadratic autonomous system. Nonlinear Dyn. **56**(4), 453–462 (2009)
7. Dadras, S., Momeni, H.R., Qi, G.: Analysis of a new 3D smooth autonomous system with different wing chaotic attractors and transient chaos. Nonlinear Dyn. **62**(1–2), 391–405 (2010)
8. Zhang, X., Zhu, H., Yao, H.: Analysis of a new three-dimensional chaotic system. Nonlinear Dyn. **67**(1), 335–342 (2012)
9. Zhang, J., Tang, W.: A novel bounded 4D chaotic system. Nonlinear Dyn. **67**(4), 2455–2465 (2012)
10. Chen, G.: Control and anticontrol of chaos. In: IEEE Proceedings on Control of Oscillations and Chaos, vol. 2, pp. 181–186 (1997)
11. Sanjuán, M.A.F., Grebogi, C.: Recent Progress in Controlling Chaos. Series on Stability, Vibration and Control of Systems, Serie B, vol. 9. World Scientific, Singapore (2010)
12. Wang, X.F.: Generating chaos in continuous-time systems via feedback control. In: Chen, G., Yu, X. (eds.) Chaos Control. Lecture Notes in Control and Information Sciences, vol. 292, pp. 179–204. Springer, Berlin (2003)
13. Zhang, H., Liu, D., Wang, Z.: Controlling Chaos: Suppression, Synchronization and Chaotification. Springer, Dordrecht (2009)
14. Bouali, S.: Feedback loop in extended van der Pol's equation applied to an economic model of cycles. Int. J. Bifurc. Chaos **9**(4), 745–756 (1999)

15. Bouali, S., Buscarino, A., Fortuna, L., Frasca, M., Gambuzza, L.V.: Emulating complex business cycles by using an electronic analogue. Nonlinear Anal., Real World Appl. **13**(6), 2459–2465 (2012)
16. Volterra, V.: Variazioni e fluttuazioni del numero d'individui in specie animali conviventi. Mem. R. Accad. Naz. dei Lincei, Ser. VI 2 (1926)
17. Lotka, A.J.: Elements of Physical Biology. Williams & Wilkins, Baltimore (1925)
18. Redheffer, R.: Lotka–Volterra systems with constant interaction coefficients. Nonlinear Anal. **46**, 1151–1164 (2001)
19. Arena, P., Baglio, S., Fortuna, L., Manganaro, G.: State controlled CNN: A new strategy for generating high complex dynamics. IEICE Trans. Fundam. Electron. Commun. Comput. Sci. **79**(10), 1647–1657 (1996)